NON-ABELIAN GAUGE THEORIES

QED does not include any selfinteraction of the photon, if one is restricted to dim ≤ 4 operators.

Yang-Mills is, instead, a theory with interacting massless vectors.

QED is associated with a $U(1)$ group, since in the case of a constant transformation, the Lagrangian is invariant under a global $U(1)$. This hints the easiest path towards constructing a non-abelian gauge theory.

Let's start with $SU(2)$. The fermion field is a doublet

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

so that under a $SU(2)$ transformation,

$$\psi(x) \longrightarrow \psi'(x) = \exp\left(\frac{-i\vec{\sigma}\cdot\vec{\theta}}{2}\right)\psi(x)$$

with σ's being the Pauli matrices. Writing $\tau^A \equiv \frac{\sigma^A}{2}$,

$$[\tau^A, \tau^B] = i\epsilon^{ABC}\tau^C$$

and θ_i are the $SU(2)$ transformation parameters.

We can start with the free Lagrangian

$$\mathcal{L}_0 = \bar{\Psi}(x)(i\not{\partial} - m)\Psi(x)$$

is invariant under the global $SU(2)$. Under the local

$$\psi(x) \longrightarrow \psi'(x) = U(\theta)\psi(x)$$

$$; \; U(\theta) = e^{-i\vec{\tau}\cdot\vec{\theta}(x)}$$

one gets that the derivative term transforms as

$$\bar{\Psi}(x)\partial_\mu\psi(x) \rightarrow \bar{\Psi}'(x)\partial_\mu\psi'(x) = \bar{\Psi}(x)\partial_\mu\psi(x)$$
$$+\bar{\Psi}(x)U^{-1}(\theta)\left(\partial_\mu U(\theta)\right)\psi(x)$$

so there is an extra term. In parallel to the abelian case, we introduce a set of vector fields in order to define the covariant derivative:

$$D_\mu\psi = (\partial_\mu - i\tau^a A^a_\mu)\psi$$

We demand that this transforms as ψ:

$$D_\mu\psi \longrightarrow (D_\mu\psi)' = U(\theta)D_\mu\psi.$$

This fixes the transformation of A^a_μ:

$$(\partial_\mu - i\,\tau^a A^{a'})U(\theta)\psi \stackrel{!}{=} U(\theta)(\partial_\mu - i\tau^a A^a_\mu)\psi$$

$$\rightarrow \left[\partial_\mu U(\theta) - i\tau^a A' U(\theta)\right]\psi = -i\,U(\theta)\,\tau^a A^a_\mu\,\psi$$

$$\rightarrow \quad \tau^a A^{a'} = U(\theta)\,\tau^a A^a_\mu\,U^{-1}(\theta) - i\left[\partial_\mu U(\theta)\right]U^{-1}(\theta)$$

First, note that this is consistent with the $U(1)$ case, for which $U = e^{i\alpha(x)}$.

For an infinitesimal transformation,

$$U(\theta) \simeq 1 - i\,\vec{\tau}\cdot\vec{\theta}(x)$$

the transformation becomes

$$\tau^a A_\mu^{a\prime} = \tau^a A_\mu^a - i\,\theta^b A_\mu^a\,[\tau^b, \tau^a] - \tau^a \partial_\mu \theta^a$$

$$= \tau^a A_\mu^a + \epsilon^{abc}\,\tau^a \theta^b A_\mu^c - \tau^a \partial_\mu \theta^a$$

or

$$A_\mu^{a\prime} = A_\mu^a + \epsilon^{abc}\,\theta^b A_\mu^c - \partial_\mu \theta^a$$

The second term tells us that A_μ^a transforms as a triplet of $SU(2)$. So the A_μ's indeed are charged under the 'gauge group', and we should therefore expect interactions.

We can construct the field strength by considering

$$[D_\mu, D_\nu]\psi = (D_\mu D_\nu - D_\nu D_\mu)\psi$$

with

$$D_\mu D_\nu \psi = \partial_\mu D_\nu \psi - i A_\mu D_\nu \psi$$

$$= \partial_\mu \partial_\nu \psi - i\,\underset{\substack{\parallel \\ -i(\partial_\mu A_\nu)\psi\, -\, i A_\nu \partial_\mu \psi}}{\partial_\mu (A_\nu \psi)} - \underbrace{i A_\mu \partial_\nu \psi}_{} - A_\mu A_\nu \psi$$

& symm.

therefore,

$$[D_\mu, D_\nu]\psi = -i(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu])\psi$$
$$= -i F_{\mu\nu}\psi$$

In components, the field strength is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu$$

From the properties of the covariant derivative we can infer how the F transforms,

$$F'_{\mu\nu} = U(\theta) F_{\mu\nu} U^{-1}(\theta)$$

therefore, unlike the abelian case, the field strength transforms, like a triplet under $SU(2)$.

The gauge invariant object is

$$tr(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{2}\delta^{ab} F^a_{\mu\nu} F^{b\mu\nu} = \frac{1}{2} F^a_{\mu\nu} F^{a\mu\nu}$$

As a summary,

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu$$
$$D_\mu\psi = (\partial_\mu - ig\tau^a A^a_\mu)\psi$$

is invariant under

$$\psi(x) \to \psi'(x) = e^{-i\vec{\tau}\cdot\vec{\theta}(x)}\psi(x) = U(\theta)\psi(x)$$

$$\tau^a A^a_\mu \to \tau^a A'^a_\mu = U(\theta)\tau^a A^a_\mu U^{-1}(\theta) - i(\partial_\mu U(\theta))U^{-1}(\theta)$$

The generalization to larger groups is straightforward, with $\tau^a \to T^a$, $\epsilon^{abc} \to f^{abc}$ etc.

A few remarks:

- Energy positivity requires G to be compact.

The kinetic term is of the form

$$tr(F_{\mu\nu}F^{\mu\nu}) \propto tr(T^A T^B) = M^{AB}$$

Energy positivity requires $M > 0$, i.e. group is compact, which is guaranteed as long as generators are hermitian.

Thm: positive definite iff group is a product of simple groups:

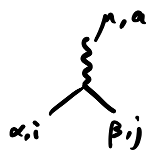
$$SU(N),\ SO(N),\ Sp(N) + 5 \text{ exceptional}$$
$$G_2, F_4, E_6, E_7, E_8.$$

We focus on $SU(N)$ since it is what appears in the SM.

For $SU(N)$ generators in the fundamental rep are $N \times N$ complex matrices T^A,

$$\begin{array}{ccc} T^{A\dagger} = T^A & , & \operatorname{tr} T^A = 0 \\ \uparrow & & \uparrow \\ M^\dagger M = 1 & & \det M = 1 \end{array}$$

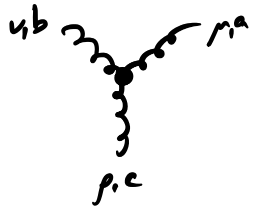
- QCD Feynman rules



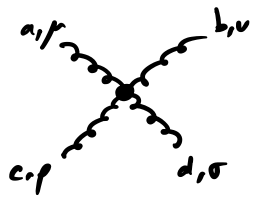
$$= ig\,(\gamma^\mu)^\alpha{}_\beta\,(T^a)^i{}_j$$

$$= \frac{-i\eta^{\mu\nu}\delta^{ab}}{p^2 + i\varepsilon}$$

$$= \frac{i(\not{p}+m)^\alpha{}_\beta}{p^2 - m^2 + i\varepsilon}$$



$$= g f^{abc}\left[\eta^{\mu\nu}(p_2-p_1)^\rho + \eta^{\nu\rho}(p_3-p_2)^\mu + \eta^{\rho\mu}(p_1-p_3)^\nu\right]$$



$$\begin{aligned} = -ig^2 \big[& f^{abe} f^{cde} (\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ & + f^{ace} f^{bde} (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ & + f^{ade} f^{bce} (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}) \big] \end{aligned}$$

- g is the Yang-Mills coupling constant, equivalent to the electromagnetic e.
- the T^a is the "charge" of the matter under the

nonabelian gauge group, given by the representation.

- Note that the same g appears for matter and for gluon "self-interaction".

Given that there is no free parameter in the three & four gluon vertices, Yang-Mills is an intrinsically interacting theory. Even in the absence of matter, YM is an interesting interacting theory.

- f^{abc} is totally antisymmetric. Therefore, there is no gluon self-interaction, but a coupling among the different N^2-1 gluons.

- These Feynman rules work at tree level. The problem is that gluon selfinteractions couple to unphysical polarizations. We need a heavier technology to deal with loops.